CONTROLLABILITY CONDITIONS OF DISTRIBUTED PARAMETER SYSTEMS WITH SMALL DAMPING

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Introduction

Problems of controllability and motion planning of distributed parameter systems have been addressed in a number of monographs and research papers (see, e.g., [1-4]). On the one hand, the question of the approximate controllability of a linear time-invariant system on a Hilbert space can be formulated in terms of an invariant subspace of the corresponding adjoint semigroup [4]. On the other hand, the problem of an effective control design remains challenging for a wide range of mechanical systems with distributed parameters.

An approach for solving the steering problem was introduced in [8] based on the optimal controls for finite dimensional subsystem. This approach was used in [9] for proving the approximate controllability of the rotating Kirchhoff plate model on an invariant manifold.

The goal of this work is to propose a constructive control strategy, based on a reduced model, and to justify that this approach can be used to solve the approximate controllability problem for the whole infinite dimensional system with damping.

This paper is organized as follows. In Section 1, the approximate controllability problem is set up and the basic controllability conditions are formulated. These controllability conditions are applied to the system representing infinite dimensional family of oscillators with damping in Section 2. Such a family of oscillators represents a rotating body-beam system.

1. Problem Statement

This paper addresses the problem of approximate controllability of a linear differential equation

\[ \dot{x} = Ax + Bu, \quad x \in H, \quad u \in \mathbb{R}^m, \]

where \( H \) is a Hilbert space, \( A: D(A) \rightarrow H \) is a closed densely defined operator, and \( B: \mathbb{R}^m \rightarrow H \) is a continuous operator. We assume that \( A \) generates a strongly continuous semigroup of operators \( \{e^{tA}\}_{t \geq 0} \) on \( H \).

Hence, for any \( x^0 \in H \) and \( u \in L^2(0, \tau) \), the mild solution of (1) corresponding to the initial condition \( x|_{t=0} = x^0 \) and control \( u = u(t) \) can be written as follows

\[ x(t; x^0, u) = e^{tA}x^0 + \int_0^t e^{(t-s)A}Bu(s)ds, \quad 0 \leq t \leq \tau. \] (2)

Recall that that system (1) is approximately controllable in time \( \tau > 0 \) if [2], given \( x^0, x^1 \in H \) and \( \varepsilon > 0 \), there exists \( u \in L^2(0, \tau) \) such that

\[ \|x(\tau; x^0, u) - x^1\| < \varepsilon. \]

In order to study the approximate controllability of system (1), we use the following result.
Proposition 1. [8] Let \( \{Q_N\} \), \( N \geq 1 \), be a family of bounded linear operators on \( H \) satisfying the following conditions:

1) \( \lim_{N \to \infty} \|Q_N x\| = 0 \), for all \( x \in H \);
2) the operators \( e^{tA} \) and \( Q_N \) commute;
3) for each \( x_0, x_1 \in H, N \geq 1 \), there is a control \( u_{x_0, x_1}^N \in L^\infty(0, \tau) \) such that

\[
(I - Q_N) (x(t); x_0^N, u_{x_0, x_1}^N) - x(t) = 0,
\]

(3)

\[
\lim_{N \to \infty} \|Q_N B \| \left\{ u_{x_0, x_1}^N \right\} _{L^2(0, \tau)} = 0.
\]

(4)

Then system (1) is approximately controllable in time \( \tau \), and the above family of functions \( u = u_{x_0, x_1}^N(t), 0 \leq t \leq \tau \), can be used to solve the approximate controllability problem.

In this paper, \( I \) stands for the identity operator on \( H \). For a possible application of the above proposition, we assume that each operator \( P_N = I - Q_N \) is a finite dimensional projection.

Let \( \dim(\text{Im} P_N) = d_N \). For given \( x_0, x_1 \in H \), we introduce vectors

\[
\tilde{x}_N^0 = P_N x_0, \quad \tilde{x}_N^1 = P_N x_1, \quad \tilde{x}_N = P_N x
\]

and operators \( \tilde{A}_N = P_N A, \quad \tilde{B}_N = P_N B \).

Then condition (3) implies that \( u = u_{x_0, x_1}^N(t) \) should solve the following control problem:

\[
\tilde{x}_N = \tilde{A}_N \tilde{x}_N + \tilde{B}_N u, \quad t \in [0, \tau],
\]

(5)

\[
\tilde{x}_N \big|_{t=0} = \tilde{x}_N^0, \quad \tilde{x}_N \big|_{t=\tau} = \tilde{x}_N^1.
\]

Here we have used the assumption that \( P_N \) and \( A \) commute as well as the property \( P_N = P_N^2 \) of a projection. To satisfy condition (4), it is natural to look for a control \( u = u_{x_0, x_1}^N(t) \) that minimizes the functional

\[
J = \int_0^\tau (Qu(t), u(t))dt
\]

(6)

with some symmetric positive definite \( m \times m \)-matrix \( Q \). As control system (5) evolves on a real \( d_N \)-dimensional vector space \( \text{Im} P_N \), we may treat (5) as a system on \( R^{d_N} \) without lack of generality. By applying the Pontryagin maximum principle, we get the optimal control for (5)-(6):

\[
\tilde{u}(t) = Q^{-1} B' e^{(\tau-t)\tilde{A}_N} \nu, \quad \nu = \int_0^\tau e^{\tilde{A}_N s} \tilde{B}_N Q^{-1} B N e^{\tilde{A}_N s} ds \left( \tilde{x}_N^1 - e^{\tilde{A}_N \tau} \tilde{x}_N^0 \right),
\]

(7)

where the prime stands for the transpose. Proposition 1 implies that the proof of the approximate controllability can be reduced to the check of condition (4) with a family of smooth controls \( u_{x_0, x_1}^N(t) = \tilde{u}(t) \) given by (7). The main contribution of this paper is the application of such a scheme for a class of systems (1) representing the oscillations of a flexible structure with damping.

2. Rotating Body-Beam System

Following [6, 7], we consider a model of a rotating body-beam system:
\[
\frac{\partial^2 w(x,t)}{\partial t^2} + c^2 \frac{\partial^4 w(x,t)}{\partial x^4} + 2\kappa \frac{\partial w(x,t)}{\partial t} = (x+d)u \, , \quad x \in (0,l), \quad (8)
\]

Here \( w(x,t) \) is the displacement of the centerline of the beam with respect to the spatial variable \( x \in [0,l] \) at time \( t \geq 0 \), \( l>0 \) is the length of the beam, \( 2\kappa>0 \) represents the viscous damping in the beam, \( c^2=EI/\rho \), \( EI \) is the flexural rigidity per unit length, and \( \rho \) is the mass per unit length of the beam. We assume that the beam is attached at the distance \( d \) from the axis of rotation of the body and that the control \( u \) is the angular acceleration of the body. The functions \( \xi_0(t) \) and \( \eta_0(t) \) are, respectively, the rotation angle and angular velocity of the rigid body.

The procedure of deriving the equations of motion is described in paper [7] for a system of several Euler-Bernoulli beams without damping. To write the equations of motion with modal coordinates, we apply Galerkin’s method to system (8).

Let \( v_n(x) \) and \( \lambda_n \) \((n=1,2,...)\) be eigenfunctions and eigenvalues of the following Sturm-Liouville problem

\[
\frac{d^4}{dx^4} v_n(x) = \lambda_n v_n(x) \, , \quad x \in (0,l) ,
\]

\[
v_n(0) = v'_n(0) = v''_n(l) = v'''_n(l) = 0 \, , \quad \|v_n\|_{L^2(0,l)} = 1. \quad (9)
\]

It is a well-known fact (cf. [5]) that \{\( v_n(x) \)\} forms an orthonormal basis in \( L^2(0,l) \) and that

\[
0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n = O(n^4) \quad \text{as} \quad n \to \infty. \quad (10)
\]

Let us fix an \( N \geq 1 \) and construct a Galerkin approximation by plugging

\[
w_N(x,t) = \sum_{j=1}^{N} v_j(x)q_j(t)
\]

into (8) and assuming that

\[
\left( \frac{\partial^2 w_N}{\partial t^2} + c^2 \frac{\partial^4 w_N}{\partial x^4} + 2\kappa \frac{\partial w_N}{\partial t}, v_n \right)_{L^2(0,l)} = \left( (x+d)u, v_n \right)_{L^2(0,l)} \quad (11)
\]

for each \( n=1,2,...,N \) and \( t \geq 0 \). As \( (v_j,v_n)_{L^2(0,l)} = \delta_{jn} \), formulae (9) and (11) imply

\[
q_n = -2\kappa q_n - c^2 \lambda_n q_n + b_n u ,
\]

\[
(12)
\]

where

\[
b_n = (x+d,v_n)_{L^2(0,l)} .
\]

\[
(13)
\]

The derivation of (12) is carried out for an arbitrary \( N \geq 1 \) and \( n \leq N \), hence, we may consider equation (12) for all \( n=1,2,... \). By exploiting this idea, we come from system (8) to the following sequence of ordinary differential equations:

\[
\dot{\xi}_0(t) = \eta_0(t) \, , \quad \eta_0(t) = u \, , \quad (14)
\]

\[
\dot{\xi}_n(t) = \omega_n \eta_n(t) \, , \quad \eta_n(t) = -\omega_n \dot{\xi}_n(t) - 2\kappa \eta_n(t) + b_n u \, , \quad n=1,2,\ldots
\]

where

\[
\xi_n = c\sqrt{\lambda_n} q_n \, , \quad \eta_n = \dot{q}_n \, , \quad \omega_n = c \sqrt{\lambda_n} > 0 \, .
\]

\[
(15)
\]

Note that representations (10) and (15) yield the following property...
\[
\sum_{i\neq j} \frac{1}{(\omega_i - \omega_j)^2} < \infty. \tag{16}
\]

Coefficients \(\omega_n\) and \(b_n\) are, respectively, the modal frequency and the control coefficient corresponding to the \(n\)-th mode of oscillations of the beam. In the sequel, we treat (14) as the model of a rotating body-beam system in Hilbert space \(l^2\), whose elements are infinite vectors

\[
x = (\xi_0, \eta_0, \xi_1, \eta_1, \xi_2, \eta_2, \cdots), \quad |x| = \left( \sum_{n=0}^{\infty} (\xi_n^2 + \eta_n^2) \right)^{1/2}.
\]

Proposition 1 implies the following controllability condition.

**Proposition 2.** Assume that the coefficients of control system (14) are given by formulae (13), (15). Then there exists a \(\tau > 0\) such that system (14), or equivalently system (1), is approximately controllable in time \(\tau\) provided that the damping coefficient \(\kappa > 0\) is small enough. The approximate controllability problem can be solved by controls of form (7) under these conditions.

**Conclusions**

This work extends the result of [8] for the case of a flexible system with damping. As it was shown earlier, condition (16) is satisfied for the Euler-Bernoulli beam without damping. Hence, condition (16) is sufficient for the approximate controllability in both conservative (\(\kappa = 0\)) and dissipative (small \(\kappa > 0\)) cases under our assumptions. An open question is whether it is possible to relax restrictions (16) in order to justify the relevance of controls (7) under a weaker assumption on the distribution of the modal frequencies \(\{\omega_n\}\).

**References**